Cohomology of regular differential forms for affine curves

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Abstract

Let C be a complex affine reduced curve, and denote by $H^1(C)$ its first truncated cohomology group, i.e. the quotient of all regular differential 1-forms by exact 1-forms. First we introduce a nonnegative invariant $\mu'(C, x)$ that measures the complexity of the singularity of C at the point x, and we establish the following formula:

$$dim\ H^1(C) = dim\ H_1(C) + \sum_{x \in C} \mu'(C, x)$$

where $H_1(C)$ denotes the first singular homology group of C with complex coefficients. Second we consider a family of curves given by the fibres of a dominant morphism $f : X \to \mathbb{C}$, where X is an irreducible complex affine surface. We analyse the behaviour of the function $y \mapsto dim\ H^1(f^{-1}(y))$. More precisely we show that it is constant on a Zariski open set, and that it is lower semi-continuous in general.

1 Introduction

Let C be a reduced complex affine curve that may be reducible or singular. For any integer k, denote by $\Omega^k(C)$ the space of regular differential k-forms (or Kähler forms) on C. The exterior derivative d is well-defined on $\Omega^k(C)$, and yields a complex:

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0(C) \longrightarrow \Omega^1(C) \longrightarrow 0$$

The first truncated De Rham cohomology group $H^1(C)$ is the quotient $\Omega^1(C)/d\Omega^0(C)$. If C is smooth, then C is a non-compact Riemann surface, for which the De Rham cohomology groups $H^k_{DR}(C)$ with complex coefficients are well-defined. Moreover $H^1(C)$ coincides with the algebraic De Rham cohomology group of C (see [Ei]) and, by a theorem of Grothendieck (see [Gro]), we have the isomorphism:

$$H^1(C) \simeq H^1_{DR}(C)$$

So truncated De Rham cohomology is always defined and coincides with standard De Rham cohomology if C is smooth. We would like to know to what extend this cohomology reflects the topological properties of C, especially when C has singularities.

Definition 1.1 *Let $\widehat{\Omega^k_{C,x}}$ be the space of formal differential k-forms on the germ (C, x). The local De Rham cohomology group of C at x is the quotient:*

$$H^1(C, x) = \widehat{\Omega^1_{C,x}}/d\widehat{\Omega^0_{C,x}}$$

Its dimension $\mu'(C, x)$ is the local Betti number of C at x.

This number characterizes the presence of singularities, in the sense that $\mu'(C, x) = 0$ if and only if x is a smooth point of C. Moreover it coincides with the Milnor number (see [Mi]) if C is locally a complete intersection (see [B-G]).

Let $H_1(C)$ be the first singular homology group of C with complex coefficients. By the results of Bloom and Herrera (see [Bl-H]), the integration of differential 1-forms along 1-cycles is well-defined and provides us with a bilinear pairing $<, >$ on $H^1(C) \times H_1(C)$ given by:

$$< \omega, \gamma >= \int_\gamma \omega$$

This induces the so-called De Rham morphism $\beta : H^1(C) \to H_1(C)^*$, $\omega \mapsto < \omega, . >$.By Poincaré Duality and a theorem of Grothendieck (see [Gro]), this map is an isomorphism when C is smooth. In the general case, we establish the following formula.

Theorem 1.2 *For any complex affine curve C, we have: dim $H^1(C) =$ dim $H_1(C) + \sum_{x\in C} \mu'(C, x)$.*

The idea of the proof is the following. For any affine curve C, the morphism β is onto (see [Bl-H]) and this yields the exact sequence:

$$0 \longrightarrow \ker \beta \longrightarrow H^1(C) \longrightarrow H_1(C)^* \longrightarrow 0$$

For any point x in C, every regular 1-form ω can be seen as a formal 1-form on the germ (C, x). Moreover every exact 1-form on C is exact as a formal 1-form on (C, x). We then have a natural morphism:

$$i_x : H^1(C) \longrightarrow H^1(C, x)$$

We prove that the morphism α:

$$\alpha : \ker \; \beta \longrightarrow \oplus_{x\in C} H^1(C, x) \quad , \quad \omega \longmapsto (i_x(\omega))_{x\in C}$$

is an isomorphism, which gives the result by passing to the dimensions.

So local Betti numbers measure the default to Poincaré Duality in the case of singular curves. Theorem 1.2 implies in particular that *a complex affine curve is isomorphic to a*

disjoint union of copies of $\mathbb{C}$ *if and only if* $H^1(C) = 0$.

Now we are going to study the behaviour of the function $h_1(y) = dim\ H^1(f^{-1}(y))$, where X is a complex affine irreducible surface and $f : X \longrightarrow \mathbb{C}$ is a dominant morphism. *The following results still hold for any reduced surface X (that is, any equidimensional reduced affine variety of dimension 2) as soon as the morphism f is dominant on every irreducible component of X*. Recall that $\mathcal{P}$ holds for every generic point of $\mathbb{C}$ if the set of points y of $\mathbb{C}$ where $\mathcal{P}(y)$ does not hold is finite. We have the following first result.

Proposition 1.3 *Let X be a complex affine irreducible surface and $f : X \to \mathbb{C}$ a dominant morphism. Then there exists an integer $h_f \geq 0$ such that, for every generic point y of $\mathbb{C}$:*

$$dim\ H^1(f^{-1}(y)) = h_f$$

For the proof, we introduce the truncated relative cohomology group $H^1(f)$ of f. We first show that it is finitely generated after a suitable localisation. This is analogous to what happens for algebraic relative cohomology groups (see [Ha], and [A-B] in the smooth case). If h_f denotes the rank of $H^1(f)$ as a $\mathbb{C}[f]$-module, we then show that it coincides with the dimension of $H^1(f^{-1}(y))$ for generic y.

Theorem 1.4 *Let X be a complex affine surface that is locally a complete intersection and $f : X \to \mathbb{C}$ be a dominant morphism. If $f^{-1}(y) \cap Sing(f)$ is finite, then:*

$$dim\ H^1(f^{-1}(y)) \leq h_f$$

In particular the function h^1 is lower semi-continuous at every point y_0 of $\mathbb{C}$ such that $f^{-1}(y_0) \cap Sing(f)$ is finite, i.e:

$$h^1(y_0) \leq \varliminf_{y \to y_0} h^1(y)$$

The previous results have analogous settings in terms of singular homology. Indeed if X is equal to $\mathbb{C}^2$ and $f : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial mapping, then there exists a non-empty Zariski open set U in $\mathbb{C}$ such that $f : f^{-1}(U) \to U$ is a locally trivial topological fibration (see [V]). In particular all the fibres $f^{-1}(y)$, $y \in U$, are homeomorphic and there exists an integer p such that:

$$dim\ H_1(f^{-1}(y)) = p$$

for any y in U. If f has isolated singularities, then its local Betti number at any singular point coincides with its Milnor number. If f is moreover tame, then every fibre $f^{-1}(y)$ has the homotopy type of a bouquet of $\mu - \mu^y$ circles, where μ is the sum of all Milnor numbers and μ^y is the sum of the Milnor numbers of critical points lying on $f^{-1}(y)$ (see [Bro]). In particular this implies that:

$$dim\ H_1(f^{-1}(y)) = \mu - \mu^y$$

By theorem 1.2, the dimension of $H^1(f^{-1}(y))$ is equal to μ if f is tame, hence it is independent of y. If f is not tame, then the dimension of $H_1(f^{-1}(y))$ may be $< \mu - \mu_y$. This loss in the topology can be interpreted as follows. The map f possesses singularities at infinity where a certain number of cycles of the general fibre vanish (see [S-T]). In this case, we see that the function $h^1(y)$ is lower semi-continuous, as asserted by theorem 1.4.

We end up this paper with an example of a mapping $f : X \to \mathbb{C}$, where X is not locally a complete intersection. In this example, the dimension of $H^1(f^{-1}(y))$ increases for a special fibre, which is contrary to what is predicted by theorem 1.4. This phenomenon is due to the presence of a special singularity on X, which produces a local Betti number for $f^{-1}(0)$ where there should be none.

2 Properties of the normalisation

Let C be a complex affine curve and $\mathcal{O}_C$ its ring of regular functions. Let $\widetilde{C}$ be its affine normalisation and $\Pi : \widetilde{C} \to C$ the normalisation morphism. Assume that $\widetilde{C}$ is embedded in $\mathbb{C}^n$, and let $\overline{B(0,R)}$ be the closed ball of $\mathbb{C}^n$ for the standard hermitian metric. For R large enough, the intersection $S = \widetilde{C} \cap \overline{B(0,R)}$ contains all the preimages of singular points of C and is a deformation retract of $\widetilde{C}$. We fix R together with a triangulation $\widetilde{T}$ of S. Let $\widetilde{V}$ be its set of vertices. Since Π is finite, we may refine $\widetilde{T}$ so that the set $V = \Pi(\widetilde{V})$ contains all the singular points of C, and so that $\widetilde{V} = \Pi^{-1}(V)$. By construction, the image $T = \Pi(\widetilde{T})$ defines a triangulation of $\Pi(S)$. Since Π is an isomorphism from $\widetilde{C} - \widetilde{V}$ to $C - V$, the set $\Pi(S)$ is also a deformation retract of C. In particular, every 1-cycle of C can be represented by a formal sum of edges of the triangulation T. We denote by $\{\widetilde{\gamma}_i\}$ the set of edges of $\widetilde{T}$, and set $\gamma_i = \Pi(\widetilde{\gamma}_i)$. *We consider this triangulation fixed from now on.*

For any point x in C, $\mathcal{O}_{C,x}$ stands for the ring of germs of regular functions at x. Denote by $\mathcal{O}_{C,V}$ the ring of germs of regular functions at V, i.e. the direct sum:

$$\mathcal{O}_{C,V} = \oplus_{x \in V} \mathcal{O}_{C,x}$$

Let I be the vanishing ideal of the set V in C, and denote by $\widehat{\mathcal{O}_{C,V}}$ the I-adic completion of $\mathcal{O}_{C,V}$. Note that we have the isomorphism:

$$\widehat{\mathcal{O}_{C,V}} = \oplus_{x \in V} \widehat{\mathcal{O}_{C,x}}$$

A *formal function on* (C,V) is an element of $\widehat{\mathcal{O}_{C,V}}$. In a similar way, denote by $\Omega^1_{C,x}$ the space of germs of regular 1-forms on C at x, and by $\Omega^1_{C,V}$ the finite sum:

$$\Omega^1_{C,V} = \oplus_{x \in V} \Omega^1_{C,x}$$

The I-adic completion $\widehat{\Omega^1_{C,V}}$ of $\Omega^1_{C,V}$ is the set of *formal 1-forms on* (C,V). Note that we have the isomorphism:

$$\widehat{\Omega^1_{C,V}} = \oplus_{x \in V} \widehat{\Omega^1_{C,x}}$$

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We can define the sets of *formal functions and formal 1-forms on* $(\widetilde{C}, \widetilde{V})$ in exactly the same way. In this section, we are going to describe the relationships between the functions and 1-forms on $\widetilde{C}$ and C.

2.1 Formal functions

Let $\Pi^* : \mathcal{O}_C \longrightarrow \mathcal{O}_{\widetilde{C}}$ be the morphism induced by the normalisation map. After localisation at V and completion, we obtain the following injective map:

$$\widehat{\Pi_V^*} : \widehat{\mathcal{O}_{C,V}} \longrightarrow \widehat{\mathcal{O}_{\widetilde{C},\widetilde{V}}}$$

Since the germ $(\widetilde{C}, x)$ is smooth for any point x in $\widetilde{C}$, every element R of $\mathcal{O}_{\widetilde{C},\widetilde{V}}$ has a well-defined order $ord_x(R)$ at x, and thus it defines a divisor:

$$div(R) = \sum_{x \in \widetilde{V}} ord_x(R)x$$

Proposition 2.1 *Let $\widetilde{R}$ be a formal function on $(\widetilde{C}, \widetilde{V})$ that vanishes at every point of $\widetilde{V}$. Then there exists a regular function S on $\widetilde{C}$, vanishing at every point of $\widetilde{V}$, and a formal function R on (C, V) such that $\widetilde{R} = S + \widehat{\Pi_V^*}(R)$.*

In order to prove this proposition, we need the following lemma.

Lemma 2.2 *With the previous notations, there exists a divisor D on $(\widetilde{C}, \widetilde{V})$ such that, for any formal function $\widetilde{R}$ on $(\widetilde{C}, \widetilde{V})$, we have: $div(\widetilde{R}) \geq D \Rightarrow \widetilde{R} \in \widehat{\Pi_V^*}(\widehat{\mathcal{O}_{C,V}})$.*

Proof: Let A be a conductor of the normalisation, i.e. an element of $\mathcal{O}_C$ that is not a zero-divisor and such that $\Pi^*(A)\mathcal{O}_{\widetilde{C}} \subseteq \Pi^*(\mathcal{O}_C)$. After localisation at V and completion, we obtain that:

$$\widehat{\Pi_V^*}(A)\widehat{\mathcal{O}_{\widetilde{C},\widetilde{V}}} \subseteq \widehat{\Pi_V^*}(\widehat{\mathcal{O}_{C,V}})$$

Set $D = div\,\widehat{\Pi_V^*}(A)$ and let $\widetilde{R}$ be a formal function on $(\widetilde{C}, \widetilde{V})$ such that $div(\widetilde{R}) \geq D$. Then $\widetilde{R}$ is locally divisible by $\widehat{\Pi_V^*}(A)$, and the quotient $S = \widetilde{R}/\widehat{\Pi_V^*}(A)$ is a formal function on $(\widetilde{C}, \widetilde{V})$. Therefore $\widetilde{R} = \widehat{\Pi_V^*}(A)S$ belongs to $\widehat{\Pi_V^*}(\widehat{\mathcal{O}_{C,V}})$.

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Proof of Proposition 2.1: Let $\widetilde{R}$ be a formal function on $(\widetilde{C}, \widetilde{V})$. For any point x in $\widetilde{V}$, let z_x be a uniformising parameter of $\widetilde{C}$ at x defined on all of $\widetilde{C}$. Then $\widetilde{R}$ has a Taylor expansion $\sum_{k \geq 0} R_{k,x} z_x^k$ at x. For any such x, we set:

$$R_x = \sum_{k \leq n} R_{k,x} z_x^k$$

Let $\mathcal{O}_{\widetilde{C}}$ be the ring of regular functions on $\widetilde{C}$, and denote by $\widetilde{I}$ the ideal generated by I in $\mathcal{O}_{\widetilde{C}}$. Since the radical of $\widetilde{I}$ is the vanishing ideal of $\widetilde{V}$, $\widehat{\mathcal{O}_{\widetilde{C},\widetilde{V}}}$ is the $\widetilde{I}$-adic completion of

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$\mathcal{O}_{\widetilde{C}}$. So there exists a regular function S on $\widetilde{C}$, whose Taylor expansion of order n at any point x is equal to R_x. For n large enough, we have the inequality:

$$div(\widetilde{R} - S) \geq D$$

By lemma 2.2, there exists a formal function R on (C, V) such that $\widehat{\Pi_V^*}(R) = \widetilde{R} - S$.

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2.2   Formal 1-forms

Let $\Pi^* : \Omega^1(C) \to \Omega^1(\widetilde{C})$ be the morphism induced by normalisation. After localisation at V and completion, we obtain the following morphism:

$$\widehat{\Pi_V^*} : \widehat{\Omega^1_{C,V}} \longrightarrow \widehat{\Omega^1_{\widetilde{C},\widetilde{V}}}$$

In this subsection, we consider $\Omega^1(\widetilde{C})$ as a $\mathcal{O}_C$-module via the multiplication rule $(P, \omega) \mapsto \Pi^*(P)\omega$. If M is an $\mathcal{O}_C$-module and $\mathcal{M}$ is an ideal, denote by $M_{\mathcal{M}}$ its localisation with respect to $\mathcal{M}$, and by $\widehat{M_{\mathcal{M}}}$ its $\mathcal{M}$-adic completion. We are going to prove the following proposition.

Proposition 2.3 *Let ω be a formal 1-form on the germ (C, V). Then there exist a formal function R on (C, V), a regular 1-form ω_0 on C and a regular function S in $\mathcal{O}_{\widetilde{C}}$, vanishing at all points of $\widetilde{V}$, such that $\omega = dR + \omega_0$ and $\Pi^*(\omega_0) = dS$.*

Lemma 2.4 *Let R be a noetherian ring, and $L : M \to N$ a morphism of finite R-modules. Let ω be an element of N that belongs to $Im \, \widehat{L_{\mathcal{M}}}$ for any maximal ideal $\mathcal{M}$. Then ω belongs to $Im \, L$.*

Proof: First we show that ω belongs to $Im \, L_{\mathcal{M}}$ for any maximal ideal $\mathcal{M}$. Let $\{e_1, .., e_k\}$ be a set of generators of M, i.e. $M = R < e_1, .., e_k >$. After localisation and completion, we get the equalities:

$$\widehat{M_{\mathcal{M}}} = \widehat{R_{\mathcal{M}}} < e_1, .., e_k > \quad \text{and} \quad Im \, \widehat{L_{\mathcal{M}}} = \widehat{R_{\mathcal{M}}} < L(e_1), .., L(e_k) >= \widehat{Im \, L_{\mathcal{M}}}$$

Since N has finite type, the $\mathcal{M}$-adic topology on N is Hausdorff and we find:

$$Im \, L_{\mathcal{M}} = Im \, \widehat{L_{\mathcal{M}}} \cap N$$

So ω belongs to $Im \, L_{\mathcal{M}}$, and for any maximal ideal $\mathcal{M}$, there exists an element $P_{\mathcal{M}}$ of $R - \mathcal{M}$ such that $P_{\mathcal{M}}\omega$ belongs to $Im \, L$. Let I be the ideal in R generated by all the $P_{\mathcal{M}}$. We claim that $I = (1)$, so that ω belongs to $Im \, L$. Indeed if I were not equal to (1), it would be contained in a maximal ideal $\mathcal{M}_0$ by Zorn's Lemma. Since I contains $P_{\mathcal{M}_0}$, $P_{\mathcal{M}_0}$ would be contained in $\mathcal{M}_0$, hence a contradiction.

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Lemma 2.5 *Let $\widetilde{\omega}$ be an element of $\Omega^1(\widetilde{C}) \cap Im\ \widehat{\Pi_V^*}$. Then $\widetilde{\omega}$ belongs to $Im\ \Pi^*$.*

Proof: We set $M = \Omega^1(C)$, $N = \Omega^1(\widetilde{C})$ and $L = \Pi^*$. Let $\mathcal{M}$ be a maximal ideal and x the corresponding point in C. If x belongs to V, then $\widetilde{\omega}$ belongs to $Im\ \widehat{L_{\mathcal{M}}}$ by assumption. If not, then $\widetilde{\omega}$ still belongs to $Im\ \widehat{L_{\mathcal{M}}}$ because x is a smooth point of C, and then $\widehat{L_{\mathcal{M}}}$ is an isomorphism. By lemma 2.4, $\widetilde{\omega}$ belongs to $Im\ \Pi^*$.

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Lemma 2.6 *Under the previous assumptions, $dim\ ker\Pi^*$ is finite and the natural map $ker\Pi^* \to ker\widehat{\Pi_V^*}$ is an isomorphism.*

Proof: For any x in C, denote by $\mathcal{M}$ the vanishing ideal of x and set $L = \Pi^*$. For any x outside V, Π is an isomorphism over an open neighborhood of x. So the map $\widehat{L_{\mathcal{M}}}$ is an isomorphism for all x outside V, and the support of $ker\Pi^*$ is contained in V. Since V is a finite set and $ker\Pi^*$ is a finite module, $ker\Pi^*$ is an artinian module and $dim\ ker\Pi^* < \infty$. So there exists an order n such that $I^n\ ker\Pi^* = 0$, and $ker\Pi^*$ is complete for the I-adic topology. Since completion is an exact functor, we have:

$$ker\Pi^* \simeq \widehat{ker\Pi^*} \simeq ker\widehat{\Pi_V^*}$$

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Proof of proposition 2.3: Let ω be a formal 1-form on the germ (C, V). Since the germ $(\widetilde{C}, \widetilde{V})$ is a disjoint union of smooth curves, the 1-form $\widehat{\Pi_V^*}(\omega)$ is exact on each of these curves. There exists a formal function $\widetilde{R}$ on $(\widetilde{C}, \widetilde{V})$ such that:

$$\widehat{\Pi_V^*}(\omega) = d\widetilde{R}$$

By proposition 2.1, there exist a regular function S on $\widetilde{C}$, vanishing at all points of $\widetilde{V}$, and a formal function R on (C, V) such that $\widetilde{R} = S + \widehat{\Pi_V^*}(R)$. After derivation, this implies:

$$\widehat{\Pi_V^*}(\omega - d\widetilde{R}) = dS$$

By lemma 2.5 applied to $\widetilde{\omega} = dS$, there exists a regular 1-form ω_1 on C such that $\Pi^*(\omega_1) = dS$. This yields:

$$\widehat{\Pi_V^*}(\omega - d\widetilde{R} - \omega_1) = 0$$

By lemma 2.6, there exists a regular 1-form ω_2 in $ker\Pi^*$ such that $\omega - d\widetilde{R} - \omega_1 = \omega_2$. Then the 1-form $\omega_0 = \omega_1 + \omega_2$ is regular on C and satisfies the following relations:

$$\omega = d\widetilde{R} + \omega_0 \quad \text{and} \quad \Pi^*(\omega_0) = dS$$

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3 Proof of theorem 1.2

Let C be a complex reduced affine curve in $\mathbb{C}^n$, and let $\beta : H^1(C) \to H_1(C)^*$ be the map defined in the introduction. Since β is onto, it induces the following complex:

$$0 \longrightarrow ker \ \beta \longrightarrow H^1(C) \longrightarrow H_1(C)^* \longrightarrow 0$$

Moreover the inclusion of regular 1-forms into formal 1-forms at x induces a morphism:

$$\alpha : ker \ \beta \longrightarrow \oplus_{x \in C} H^1(C, x)$$

Since C carries a structure a CW-complex, the vector space $H_1(C)$ is finite dimensionnal, and the same holds for every $H^1(C, x)$ (see [B-G]). So for the proof of theorem 1.2, we only need to show that α is an isomorphism, and the result will follow by passing to the dimensions.

3.1 Injectivity of α

Without loss of generality, we may assume that the curve C is connected. Let ω be an element of $ker \ \beta$. Fix a point x_0 in C, and consider the map R defined as follows. For any point x in C, choose a path γ going from x_0 to x, and set:

$$R(x) = \int_\gamma \omega$$

Since ω has null integral along any closed path in C, this number is well-defined and independent of the path γ chosen. Furthermore the function $S = R \circ \Pi$ is holomorphic on $\widetilde{C}$ because it defines an integral of $\Pi^*(\omega)$ on $\widetilde{C}$. By Grothendieck's Theorem, S is a regular function on $\widetilde{C}$, and S takes the value $R(x)$ on $\Pi^{-1}(x)$.

Assume now that $\alpha(\omega) = 0$. Then for any point x of C, the class of ω in $H^1(C, x)$ is zero, and there exists a formal function R^x on the germ (C, x) such that $\omega = dR^x$. Let $\mathcal{M}$ be the vanishing ideal of x and denote by $\widehat{L_\mathcal{M}}$ the morphism induced by Π^* after localisation at $\mathcal{M}$ and completion. The formal function $S - \widehat{L_\mathcal{M}}(R^x)$ on $(\widetilde{C}, \Pi^{-1}(x))$ is constant around every point of $\Pi^{-1}(x)$, because S and $\widehat{L_\mathcal{M}}(R^x)$ are both integrals of $\Pi^*(\omega)$. Since S and $\widehat{L_\mathcal{M}}(R^x)$ are constant on $\Pi^{-1}(x)$, there exists a constant λ such that:

$$S - \widehat{L_\mathcal{M}}(R^x) = \lambda$$

on $(\widetilde{C}, \Pi^{-1}(x))$. Up to replacing R^x by $R^x - \lambda$, we may assume that $\lambda = 0$, and so S belongs to $Im \ \widehat{L_\mathcal{M}}$ for any point x in C. By applying lemma 2.4 to the morphism $\Pi^* : \mathcal{O}_C \to \mathcal{O}_{\widetilde{C}}$ of finite $\mathcal{O}_C$-modules, we get that S belongs to $\mathcal{O}_C$. Since $S = R^x$ for any x in C, we get by derivation:

$$\omega = dS = dR^x \quad \text{in} \quad \widehat{\Omega^1_{C,x}}$$

Since $\Omega^1_{C,x}$ is a finite $\mathcal{O}_{C,x}$-module, the $\mathcal{M}$-adic topology is separated and $\omega = dS$ in $\Omega^1_{C,x}$. By Bourbaki result (Commutative Algebra Chap 1-7 Corollary 1, p. 88), $\omega = dS$ in $\Omega^1(C)$ and the class of ω in $H^1(C)$ is zero.

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3.2 Surjectivity of α

By construction, the set V contains all the singular points of C. Since $H^1(C, x) = 0$ if C is smooth at x, we have the isomorphism:

$$\oplus_{x \in C} H^1(C, x) \simeq \oplus_{x \in V} H^1(C, x)$$

So every element ω of this sum can be represented by a formal 1-form on (C, V), which we also denote by ω. By lemma 2.3, there exist a formal function R on (C, V), a regular 1-form ω_0 on C and a regular function S on $\widetilde{C}$, vanishing at all points of $\widetilde{V}$, such that:

$$\omega = dR + \omega_0 \quad \text{and} \quad \Pi^*(\omega_0) = dS$$

Let γ be a 1-cycle in C. This cycle can be represented as a formal linear combination of the edges γ_i of the triangulation T. Since S vanishes at all vertices of $\widetilde{T}$, and these vertices are endpoints of the $\widetilde{\gamma_i}$, we have:

$$\int_{\gamma_i} \omega_0 = \int_{\widetilde{\gamma_i}} \Pi^*(\omega_0) = \int_{\widetilde{\gamma_i}} dS = S(\widetilde{\gamma_i}(1)) - S(\widetilde{\gamma_i}(0)) = 0$$

By linearity, we get that $< \omega_0, \gamma > = 0$ for any cycle γ in C. So ω_0 belongs to $ker\ \beta$ and represents the same class as ω in $\oplus_{x \in V} H^1(C, x)$. Therefore $\alpha(\omega_0) = \omega$ and α is surjective.

4 Relative cohomology

Let X be a complex irreducible affine surface, and $f : X \longrightarrow \mathbb{C}$ a dominant morphism. Denote by $\Omega^k(X)$ the space of regular k-forms on X. The *first group of truncated relative cohomology of f* is the quotient:

$$H^1(f) = \frac{\Omega^1(X)}{d\Omega^0(X) + \Omega^0(X)df}$$

Note that $H^1(f)$ is a $\mathbb{C}[f]$-module via the multiplication $(P(f), \omega) \mapsto P(f)\omega$. In the case of analytic germs f, relative cohomology groups have been extensively used to describe the topological and cohomological properties of f; for more details, see for instance [Loo]. In the algebraic setting, the relative cohomology of polynomial mappings has been intensively studied, especially via the use of the Gauss-Manin connexion (see for instance [A-B]).We are going to study some properties of truncated relative cohomology and use them to prove proposition 1.3.

4.1 Finiteness of truncated relative cohomology

In this subsection, we are going to establish that $H^1(f)$ is, after a suitable localisation, a finite module. More precisely:

Proposition 4.1 *Let $f : X \longrightarrow \mathbb{C}$ be a dominant morphism, where X is an irreducible surface. Then there exists a non-zero polynomial P of $\mathbb{C}[t]$ such that $H^1(f)_{(P(f))}$ is a $\mathbb{C}[f]_{(P(f))}$-module of finite type.*

We introduce the following $\mathbb{C}[f]$-modules M_0 and M_1:

$$M_0 = \frac{\{\omega \in \Omega^1(X), \exists \eta \in \Omega^1(X), d\omega = \eta \wedge df\}}{d\Omega^0(X) + \Omega^0(X)df}$$

$$M_1 = \frac{\Omega^1(X)}{\{\omega \in \Omega^1(X), \exists \eta \in \Omega^1(X), d\omega = \eta \wedge df\}}$$

Note that we have the exact sequence of $\mathbb{C}[f]$-modules:

$$0 \longrightarrow M_0 \longrightarrow H^1(f) \longrightarrow M_1 \longrightarrow 0$$

Since localisation is an exact functor and $\mathbb{C}[f]_{(P(f))}$ is a noetherian module for any $P \neq 0$, it suffices to prove that both M_0 and M_1 become finite modules after a suitable localisation. The module M_0 is by definition the first group of standard relative cohomology of f (see []). By a theorem of Hartshorne (see [Ha]), there exists a non-zero polynomial P of $\mathbb{C}[t]$ such that $(M_0)_{(P(f))}$ is a $\mathbb{C}[f]_{(P(f))}$-module of finite type. So there only remains to prove that M_1 becomes a finite module after a suitable localisation, and this is what we will do in the following lemmas.

Lemma 4.2 *Let X be an irreducible affine surface, S its singular set and I the defining ideal of S in $\mathcal{O}_X$. Let $f : X \to \mathbb{C}$ be a dominant map. Then there exists a non-zero polynomial P of $\mathbb{C}[t]$ such that, for any $n \geq 0$, the quotient $(\Omega^1(X)/I^n)_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module.*

Proof: Since S has dimension ≤ 1, there exists a non-empty Zariski open set U in $\mathbb{C}$ such that either $f^{-1}(U) \cap S$ is empty or the restriction $f : f^{-1}(U) \cap S \longrightarrow U$ is a finite morphism. Let P be a non-zero polynomial whose roots form the set $\mathbb{C} - U$. In the first case, $(\mathcal{O}_S)_{(P(f))}$ is equal to zero. In the second case, the ring $(\mathcal{O}_S)_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module. Since I is radical, $\mathcal{O}_S$ coincides with $\mathcal{O}_X/I$ and $(\mathcal{O}_X/I)_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module. It is then easy to prove that $(\mathcal{O}_X/I^n)_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module, by an induction on n and by using the following exact sequence:

$$0 \longrightarrow (I^n/I^{n+1})_{(P(f))} \longrightarrow (\mathcal{O}_X/I^{n+1})_{(P(f))} \longrightarrow (\mathcal{O}_X/I^n)_{(P(f))} \longrightarrow 0$$

Here the only thing to note is that I^n/I^{n+1} is a finite $\mathcal{O}_X/I$-module for any n. Since $\Omega^1(X)$ is a finite $\mathcal{O}_X$-module, $\Omega^1(X)/I^n$ is a finite $\mathcal{O}_X/I^n$-module. Therefore $(\Omega^1(X)/I^n)_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module.

■

Lemma 4.3 *Let X be an irreducible affine surface, S its singular set and I the defining ideal of S in $\mathcal{O}_X$. Let $f : X \to \mathbb{C}$ be a dominant map. Then there exists a non-zero polynomial P of $\mathbb{C}[t]$ and an integer N such that:*

$$I^N \Omega^2(X)_{(P(f))} \subseteq \Omega^1(X)_{(P(f))} \wedge df$$

Proof: By the generic smoothness theorem (see [Jou]), there exists a non-empty Zariski open set U of $\mathbb{C}$ such that the restriction $f : f^{-1}(U) \cap (X - S) \longrightarrow U$ is non-singular. Let P be a non-zero polynomial whose roots form the set $\mathbb{C} - U$, and denote by X' the surface $f^{-1}(U) \cap X$. We can then identify $\Omega^i(X)_{(P(f))}$ with $\Omega^i(X')$ for any i. We are going to prove there exists an integer N such that:

$$I^N \Omega^2(X') \subseteq \Omega^1(X') \wedge df$$

Let $f_1, ..., f_r$ be a set of generators of I. Let Ω be a regular 2-form on X'. Since f_i belongs to I, the surface $X' - V(f_i)$ is smooth and the restriction $f : X' - V(f_i) \longrightarrow U$ is non-singular. By the De Rham lemma (see []), there exists a regular 1-form η_i on $X' - V(f_i)$ such that $\Omega = \eta_i \wedge df$ on $X' - V(f_i)$. Write η_i as $\theta_i / f_i^{n_i}$, where θ_i is regular on X'. Then there exists an integer m_i such that:

$$f_i^{m_i}(f_i^{n_i}\Omega - \theta_i \wedge df) = 0$$

on X', and so $f_i^{m_i+n_i}\Omega$ belongs to $\Omega^1(X') \wedge df$. We set $N_\Omega = r \sup\{m_i + n_i\}$. Every element g of I^{N_Ω} can be written as a linear combination of the form:

$$g = \sum_{i_1 + ... + i_r = N_\Omega} a_{i_1,...,i_r} f_1^{i_1}...f_r^{i_r}$$

Since $i_1 + ... + i_r = N_\Omega$, at least one of the indices i_k is no less than $m_k + n_k$. So for any multi-index $(i_1, ..., i_r)$, $a_{i_1,...,i_r} f_1^{i_1}...f_r^{i_r}\Omega$ belongs to $\Omega^1(X') \wedge df$. Therefore $g\Omega$ belongs to $\Omega^1(X') \wedge df$ for any g in I^{N_Ω}. Now let $\Omega_1, ..., \Omega_s$ be a set of generators of $\Omega^2(X')$ as an $\mathcal{O}_{X'}$-module. If $N \geq N_{\Omega_i}$ for any $i = 1, .., s$, then we have obviously:

$$I^N \Omega^2(X') \subseteq \Omega^1(X') \wedge df$$

∎

Lemma 4.4 *Let $f : X \to \mathbb{C}$ be a dominant map, where X is an irreducible affine surface. Then there exists a non-zero polynomial P of $\mathbb{C}[t]$ such that $(M_1)_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module.*

Proof: We keep the notations of lemma 4.3. For any element ω of $I^{N+1}\Omega^1(X')$, the 2-form $\Omega = d\omega$ belongs to $I^N \Omega^2(X')$, hence to $\Omega^1(X') \wedge df$ by lemma 4.3. Therefore we have the inclusion:

$$I^{N+1}\Omega^1(X') \subseteq \{\omega \in \Omega^1(X'), \exists \eta \in \Omega^1(X'), d\omega = \eta \wedge df\}$$

By lemma 4.2, there exists a non-zero polynomial Q such that $(\Omega^1(X)/I^n)_{(Q(f))}$ is a finite $\mathbb{C}[f]_{(Q(f))}$-module. The previous inclusion then induces the following surjective morphism of $\mathbb{C}[f]_{(PQ(f))}$-modules:

$$L : (\Omega^1(X)/I^n)_{(PQ(f))} \longrightarrow (M_1)_{(PQ(f))}$$

Since $(\Omega^1(X)/I^n)_{(PQ(f))}$ is finite over $\mathbb{C}[f]_{(PQ(f))}$, the result follows.

∎

4.2   Proof of proposition 1.3

In this subsection, we are going to prove more than proposition 1.3. More precisely we are going to relate the rank of $H^1(f)$ (which is finite by proposition 4.1) to the dimension of the $H^1(f^{-1}(y))$.

Proposition 4.5 *Let $f : X \longrightarrow \mathbb{C}$ be a dominant morphism, where X is an irreducible affine surface. Let h_f be the rank of the module $H^1(f)$. Then for generic y in $\mathbb{C}$, the dimension of $H^1(f^{-1}(y))$ is equal to h_f.*

Lemma 4.6 *Let X be a complex affine surface and $f : X \longrightarrow \mathbb{C}$ a dominant morphism. If $(f - y)$ is a radical ideal in $\mathcal{O}_X$, then $H^1(f)/(f - y) \simeq H^1(f^{-1}(y))$. In particular, this holds for generic y.*

Proof: By definition, we have a first isomorphism:

$$H^1(f)/(f - y) \simeq \frac{\Omega^1(X)}{d\Omega^0(X) + \Omega^0(X)df + (f - y)\Omega^1(X)}$$

$$\simeq \frac{\Omega^1(X)/\Omega^0(X)df + (f - y)\Omega^1(X)}{d\Omega^0(X) + \Omega^0(X)df + (f - y)\Omega^1(X)/\Omega^0(X)df + (f - y)\Omega^1(X)}$$

Since $(f - y)$ is a radical ideal in $\mathcal{O}_X$, the restriction morphism induces an isomorphism:

$$\Omega^1(X)/\Omega^0(X)df + (f - y)\Omega^1(X) \simeq \Omega^1(f^{-1}(y))$$

From that we deduce $H^1(f)/(f - y) \simeq \Omega^1(f^{-1}(y))/d\Omega^0(f^{-1}(y)) = H^1(f^{-1}(y))$.

∎

Proof of proposition 4.5: Let $f : X \longrightarrow \mathbb{C}$ be a dominant morphism, where X is an irreducible affine surface. Let h_f be the rank of the module $H^1(f)$. By proposition 4.1, there exists a non-zero polynomial P of $\mathbb{C}[t]$ such that $H^1(f)_{(P(f))}$ is a finite $\mathbb{C}[f]_{(P(f))}$-module. Up to refining the localisation, we may even assume that $H^1(f)_{(P(f))}$ is a finite free $\mathbb{C}[f]_{(P(f))}$-module of rank h_f. For any y such that $(f - y)$ is a radical ideal in $\mathcal{O}_X$ and $P(y) \neq 0$, we have by lemma 4.6:

$$H^1(f)_{(P(f))}/(f - y) \simeq H^1(f)/(f - y) \simeq H^1(f^{-1}(y))$$

Since $H^1(f)_{(P(f))}$ is finite free of rank h_f, $H^1(f)_{(P(f))}/(f - y)$ has dimension h_f and the result follows.

5 The property $\mathcal{P}$

In this subsection, we are going to prove the inequality given in theorem 1.4 by using a special property of the relative cohomology group $H^1(f)$. This property will enable us to control the dimension of $H^1(f^{-1}(t))$ by means of the rank of $H^1(f)$.

Definition 5.1 *A $\mathbb{C}[f]$-module M satisfies the property $\mathcal{P}(y)$ if for any integer r and any element ω of M, we have: $(f - y)^r \omega = 0 \Longrightarrow \omega \in (f - y)M$.*

Lemma 5.2 *Let M be a $\mathbb{C}[f]$-module satisfying $\mathcal{P}(y)$. Then dim $M/(f - y) \leq rk\ M$.*

Proof: Let $e_1, .., e_s$ be some elements of M whose classes in $M/(f - y)$ are free. In order to establish the lemma, we prove by contradiction that $e_1, .., e_s$ are free in M. Assume there exist some polynomials $P_1(f), .., P_s(f)$ not all zero such that $P_1(f)e_1 + ... + P_s(f)e_s = 0$ in M. Let m be the minimum of the orders of the P_i at y. Every $P_i(f)$ can be written as $P_i(f) = (f - y)^m T_i(f)$ where at least one of the $T_i(y)$ is nonzero. So we get:

$$(f - y)^m \left\{ T_1(f)e_1 + ... + T_s(f)e_s \right\} = 0$$

By the property $\mathcal{P}(y)$, this implies:

$$T_1(f)e_1 + ... + T_s(f)e_s \equiv T_1(y)e_1 + ... + T_s(y)e_s \equiv 0\ [(f - y)]$$

Since the e_i are free modulo $(f - y)$, every $T_i(y)$ is zero, hence a contradiction.

∎

Our purpose in this subsection is to prove:

Proposition 5.3 *Let X be a complex irreducible affine surface, and $f : X \longrightarrow \mathbb{C}$ a dominant morphism. Assume that X is locally a complete intersection. If $f^{-1}(y) \cap Sing(f)$ is finite, then $H^1(f)$ satisfies the property $\mathcal{P}(y)$.*

Since X is locally a complete intersection, the finiteness of $f^{-1}(y) \cap Sing(f)$ implies that $(f - y)$ is a radical ideal in $\mathcal{O}_X$. By lemma 4.6, we have $H^1(f)/(f - y) \simeq H^1(f^{-1}(y))$. So theorem 1.4 will follow from lemma 5.2 and proposition 5.3. We begin with a few lemmas.

Lemma 5.4 *Let X be a complex affine surface that is locally a complete intersection. Let ω be a regular 1-form on X and A a regular function on X such that $(f - y)\omega = Adf$. If $f^{-1}(y) \cap Sing(f)$ is finite, there exists a regular function B on X such that $\omega = Bdf$.*

Proof: Let ω be a regular 1-form on X and A a regular function on X such that $(f - y)\omega = Adf$. Then A vanishes on the set $f^{-1}(y) - Sing(f)$. Since $f^{-1}(y)$ is equidimensionnal of dimension 1 and $f^{-1}(y) \cap Sing(f)$ is finite, A vanishes on $f^{-1}(y)$. Since $f^{-1}(y) \cap Sing(f)$ is finite and X is locally a complete intersection, $f^{-1}(y)$ defines locally a complete intersection. Hence it is a complete intersection on X, and $(f - y)$ divides A. If $A = (f - y)B$, then $(f - y)(\omega - Bdf) = 0$. Since X is locally a complete intersection, the module $\Omega^1(X)$ is torsion-free (see [Gr]) and $\omega = Bdf$.

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■

Lemma 5.5 *Let X be a complex irreducible affine surface and $f : X \to \mathbb{C}$ a dominant morphism. Let $C_1, .., C_r$ be the connected components of $f^{-1}(t)$ and n an integer ≥ 0. Then there exist some regular functions $S_{i,n}$ on X such that $S_{i,n} = 1$ on C_i, $S_{i,n} = 0$ on C_j for $j \neq i$ and $dS_{i,n}$ belongs to $(f - t)^{n+1}\Omega^1(X)$.*

Proof: For simplicity, assume that $t = 0$. There exists a regular function T_i on X such that $T_i = 1$ on C_i and $T_i = 0$ on C_j for $j \neq i$. Then $T_i(1 - T_i)$ vanishes on $f^{-1}(0)$ and by Hilbert's Nullstellensatz, there exists an integer m such that $T_i^m(1 - T_i)^m$ belongs to $f^{n+1}\mathcal{O}_X$. We set:

$$P_i(x) = \int_0^x t^m(1 - t)^m dt \quad \text{and} \quad R_{i,n} = P_i(T_i)$$

By construction the 1-form $dR_{i,n} = T_i^m(1-T_i)^m dT_i$ is divisible by f^{n+1}. Since $P_i(0) = 0$ and T_i vanishes on C_j for $j \neq i$, $R_{i,n}$ vanishes on C_j if $j \neq 0$. Since $P_i(1) \neq 0$, $R_{i,n} = P_i(1) \neq 0$ on C_i. Then choose $S_{i,n} = R_{i,n}/P_i(1)$.

■

Lemma 5.6 *Let X be a complex irreducible affine surface and $f : X \to \mathbb{C}$ a dominant morphism. Let R be a regular function on X such that $dR = Adf + (f - t)\eta$, where A, η are regular on X. Then R is locally constant on $f^{-1}(t)$.*

Proof: Since $dR = Adf + (f-t)\eta$, the restriction of dR to $f^{-1}(t)$ is zero. So R is singular at any smooth point of $f^{-1}(t)$, and R is constant on every connected component of the smooth part of $f^{-1}(t)$. By continuity and density, R is constant on every connected component of $f^{-1}(t)$, hence it is locally constant on $f^{-1}(t)$.

■

Proof of proposition 5.3: Let X be a complex irreducible affine surface that is locally a complete intersection. Let $f : X \to \mathbb{C}$ be a dominant morphism and assume that $f^{-1}(t) \cap Sing(f)$ is finite. We may assume that $t = 0$. Let us prove by induction on $n \geq 0$ that, if $f^n\omega = 0$ in $H^1(f)$, then ω belongs to $(f)H^1(f)$. This is trivial for $n = 0$. Assume that the assertion holds to the order n. Let ω be a regular 1-form on X such that $f^{n+1}\omega = 0$ in $H^1(f)$. Then there exist some regular functions R, A such that $f^{n+1}\omega = dR + Adf$ on $\Omega^1(X)$. By lemma 5.6, R is locally constant on $f^{-1}(0)$. Let $C_1, .., C_r$ be the connected components of $f^{-1}(0)$. If R takes the value λ_i on C_i, then the function:

$$R' = R - \sum_i \lambda_i S_{i,n+1}$$

vanishes on $f^{-1}(0)$. By construction, there exists a regular 1-form η such that:

$$f^{n+1}\omega = dR' + Adf + f^{n+2}\eta$$

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Since $f^{-1}(0) \cap Sing(f)$ is finite and X is locally a complete intersection, (f) is a radical ideal and R' is divisible by f. If $R' = fS$ with S regular, we obtain:

$$f\left(f^n\omega - dS - f^{n+1}\eta\right) = (A+S)df$$

By lemma 5.4, there exists a regular function B such that:

$$f^n(\omega - f\eta) = dS + Bdf$$

By induction $(\omega - f\eta)$ belongs to $(f)H^1(f)$, as well as ω, and we are done.

■

6   An example

We end this paper with an example of a surface that is not locally a complete intersection (for more details, see [Di]). For that surface there exists a map for which the conclusion of theorem 1.4 fails. Let (u, v, w_1, w_2) be a system of coordinates in $\mathbb{C}^4$, and consider the affine set X of $\mathbb{C}^4$ defined by the equations:

$$u^2 w_1 - v^2 = 0, \quad u^3 w_2 - v^3 = 0, \quad w_1^3 - w_2^2 = 0$$

Note that X can be reinterpreted as:

$$X = Spec(\mathbb{C}[x, xy, y^2, y^3])$$

So X is an irreducible surface. Moreover 0 is the only singular point of X, but X is not locally a complete intersection. Indeed if it were so, then X would be a normal surface because it is non-singular in codimension 1. Consider the function $h = w_2/w_1 = v/u$ on X. It is well-defined and regular outside the origin, hence h is regular because X is normal. Moreover we have the following relations:

$$v = hu, \qquad w_1 = h^2, \qquad w_2 = h^3$$

So every regular function on X can be expressed as a polynomial in (u, h), and X is isomorphic to $\mathbb{C}^2$. But this is impossible because X is singular at the origin. Consider now the map $f : X \to \mathbb{C}$ defined by:

$$f(u, v, w_1, w_2) = u$$

For $y \neq 0$, the fibre $f^{-1}(y)$ is isomorphic to a line, hence $H^1(f^{-1}(y)) = 0$. The fibre $f^{-1}(0)$ is isomorphic to a cusp, hence contractible, and $f^{-1}(0) \cap Sing(f)$ is reduced to the origin. Moreover its Milnor number coincides with its local Betti number and is equal to 2. With the notations of the previous sections, $h_f = 0$ and dim $H^1(f^{-1}(0)) = 2$, so that dim $H^1(f^{-1}(0)) \not\leq h_f$.

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